Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number | | | | | | Candidate Number | | | | |

Pearson Edexcel International Advanced Level

Time 1 hour 30 minutes

Paper reference **WFM01/01**

Mathematics

International Advanced Subsidiary/Advanced Level
Further Pure Mathematics F1

You must have:
Mathematical Formulae and Statistical Tables (Yellow), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

Turn over ▶

Pearson

1. $z_1 = 3 + 3\text{i} \qquad z_2 = p + q\text{i} \qquad p, q \in \mathbb{R}$

Given that $|z_1 z_2| = 15\sqrt{2}$

(a) determine $|z_2|$ **(2)**

Given also that $p = -4$

(b) determine the possible values of q **(2)**

(c) Show z_1 and the possible positions for z_2 on the same Argand diagram. **(2)**

a. $\boxed{|z_1 z_2| = |z_1|\,|z_2|}$

$15\sqrt{2} = |z_1|\,|z_2|$

find $|z_1|$ first : $|z_1| = \sqrt{(3)^2 + (3)^2} = 3\sqrt{2}$

$\therefore |z_1| = 3\sqrt{2}$

$15\sqrt{2} = 3\sqrt{2}\,|z_2|$ $\quad \div 3\sqrt{2}$

$|z_2| = \dfrac{15\sqrt{2}}{3\sqrt{2}} = 5$

$\boxed{|z_2| = 5}$

b. $|z_2| = 5$ using part (a)

$|p + q\text{i}| = 5$

$|-4 + q\text{i}| = 5$

$\sqrt{(-4)^2 + (q)^2} = 5$ $\quad$ square both sides

$\left(\cancel{\sqrt{(-4)^2 + (q)^2}}\right)^{\cancel{2}} = (5)^2$

$(-4)^2 + (q)^2 = (5)^2$

$16 + q^2 = 25$

$q^2 = 9$

$q = \pm\sqrt{9} = \pm 3$

$\boxed{q = \pm 3}$

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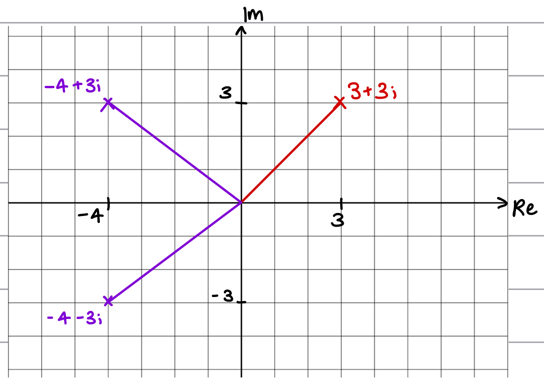
Question 1 continued

c. $z_1 = 3 + 3i$

$z_2 = -4 + 3i$ or $z_2 = -4 - 3i$



2. $$f(x) = 10 - 2x - \frac{1}{2\sqrt{x}} - \frac{1}{x^3} \qquad x > 0$$

(a) Show that the equation $f(x) = 0$ has a root α in the interval [0.4, 0.5] **(2)**

(b) Determine $f'(x)$. **(3)**

(c) Using $x_0 = 0.5$ as a first approximation to α, apply the Newton-Raphson procedure once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places. **(2)**

The equation $f(x) = 0$ has another root β in the interval [4.8, 4.9]

(d) Use linear interpolation once on the interval [4.8, 4.9] to find an approximation to β, giving your answer to 3 decimal places. **(2)**

a. $f(0.4) = 10 - 2(0.4) - \frac{1}{2\sqrt{0.4}} - \frac{1}{(0.4)^3} = -7.215569415$

$f(0.5) = 10 - 2(0.5) - \frac{1}{2\sqrt{0.5}} - \frac{1}{(0.5)^3} = 0.2928932188$

There is a sign change from negative to positive and $f(x)$ is continuous $\therefore$ a root, α, lies in [0.4, 0.5].

b. rewrite $f(x)$ as indices:

$f(x) = 10 - 2x - \frac{1}{2}x^{-\frac{1}{2}} - x^{-3}$

$f'(x) = -2 + \frac{1}{4}x^{-3/2} + 3x^{-4}$

$\boxed{f'(x) = -2 + \frac{1}{4}x^{-\frac{3}{2}} + 3x^{-4}}$

c. $X_1 = 1.75$

we are looking for X_2

Numerical Sol^n's of Eq^n's:

ewton-Rhapson iteration for solving $f(x) = 0$

$X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}$

Question 2 continued

From Newton-Rhapson iteration: $X_2 = X_1 - \frac{f(X_1)}{f'(X_1)}$

(1) must find $f(X_1)$ by subbing in X_1 (1.75) into $f(x)$

(2) must find $f'(X_1)$ by differentiating $f(x)$ and subbing in X_1 (1.75) into $f'(x)$.

(1) $f(0.5) = 0.2928932188$ from part (a)

(2) $f'(x) = -2 + \frac{1}{4}x^{-3/2} + 3x^{-4}$ from part (b)

$f'(0.5) = -2 + \frac{1}{4}(0.5)^{-3/2} + 3(0.5)^{-4} = 46.70710678$

$X_2 = (0.5) - \frac{0.2928932188}{46.70710678}$

$X_2 = 0.4937291509$

$\boxed{\alpha = 0.494 \ (3\text{d.p.})}$

d. Linear Interpolation:

$\alpha = \frac{af(b) - bf(a)}{f(b) - f(a)}$ for interval $[a, b]$

where α is the root

$a = 4.8 \quad b = 4.9$

$\beta = \frac{4.8 f(4.9) - 4.9 f(4.8)}{f(4.9) - f(4.8)}$

$f(4.8) = 10 - 2(4.8) - \frac{1}{2\sqrt{4.8}} - \frac{1}{(4.8)^3} = 0.1627400223$

$f(4.9) = 10 - 2(4.9) - \frac{1}{2\sqrt{4.9}} - \frac{1}{(4.9)^3} = -0.03437683548$

Question 2 continued

$$\beta = \frac{4.8(-0.03437683548) - 4.9(0.1627400223)}{-0.03437683548 - 0.1627400223}$$

$\beta = 4.882560175$

$\boxed{\beta = 4.883 \ (3\text{d.p.})}$

3. $\mathbf{M} = \begin{pmatrix} k & k \\ 3 & 5 \end{pmatrix}$ where k is a non-zero constant

(a) Determine $\mathbf{M}^{-1}$, giving your answer in simplest form in terms of k. **(2)**

Hence, given that $\mathbf{N}^{-1} = \begin{pmatrix} k & k \\ 4 & -1 \end{pmatrix}$

(b) determine $(\mathbf{MN})^{-1}$, giving your answer in simplest form in terms of k. **(2)**

a. $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(X) = (a)(d) - (b)(c)$

$\det(M) = (k)(5) - (k)(3)$

$= 5k - 3k = 2k$

$\det(M) = 2k$

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad X^{-1} = \dfrac{1}{\det(X)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$-b = -(k) = -k$

$-c = -(3) = -3$

$\boxed{M^{-1} = \dfrac{1}{2k} \begin{bmatrix} 5 & -k \\ -3 & k \end{bmatrix}}$

b. $(MN^{-1}) = N^{-1}M^{-1}$

$(MN^{-1}) = \begin{bmatrix} k & k \\ 4 & -1 \end{bmatrix} \times \dfrac{1}{2k} \begin{bmatrix} 5 & -k \\ -3 & k \end{bmatrix}$ can use M^{-1} from part (a)

$= \dfrac{1}{2k} \begin{bmatrix} k & k \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 5 & -k \\ -3 & k \end{bmatrix}$

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Question 3 continued

$$= \frac{1}{2k}\begin{bmatrix} (k)(5)+(k)(-3) & (k)(-k)+(k)(k) \\ (4)(5)+(-1)(-3) & (4)(-k)+(-1)(k) \end{bmatrix}$$

$$= \frac{1}{2k}\begin{bmatrix} 2k & 0 \\ 23 & -5k \end{bmatrix}$$

$$\boxed{(MN)^{-1} = \frac{1}{2k}\begin{bmatrix} 2k & 0 \\ 23 & -5k \end{bmatrix}}$$

(Total for Question 3 is 4 marks)

4. $$f(z) = 2z^4 - 19z^3 + Az^2 + Bz - 156$$

where A and B are constants.

The complex number $5 - i$ is a root of the equation $f(z) = 0$

(a) Write down another complex root of this equation.

(1)

(b) Solve the equation $f(z) = 0$ completely.

(5)

(c) Determine the value of A and the value of B.

(2)

a. For a quartic eq^n with form: $ax^4 + bx^3 + cx^2 + dx + e$, there are 4 roots.
There are 3 possibilities: 4 real roots
2 real roots and 2 complex roots (conjugate pair)
4 complex roots (2 conjugate pairs).

The other complex root must be the complex conjugate of $(5-i)$
(flip sign of imaginary component

$\boxed{5+i}$

b. $(z-(5+i))(z-(5-i))(az^2+bz+c)$
$(z-5-i)(z-5+i)(az^2+bz+c)$
$(z^2-5z \cancel{+iz} - 5z + 25 \cancel{-5i} \cancel{-iz} \cancel{+5i} - i^2)(az^2+bz+c)$
$(z^2-10z+25-(-1))(az^2+bz+c)$
$(z^2-10z+26)(az^2+bz+c)$
$az^4+bz^3+cz^2-10az^3-10bz^2-10cz+26az^2+26bz+26c$
$(a)z^4 + (b-10a)z^3 + (c-10b+26a)z^2 + (-10c+26b)z + (26c)$

equate coefficients to $f(z)$: (1) $a = 2$
(2) $b - 10a = -19$
(3) $c - 10b + 26a = A$
(4) $-10c + 26b = B$
(5) $26c = -156$

(1) $a = 2$ (5) $26c = -156$
$c = \frac{-156}{26}$
$c = -6$

Question 4 continued

sub $a=2$ into (2) and solve b.

(2) $b-10a=-19$

$b-10(2)=-19$

$b-20=-19$

$b=1$

$\therefore$ quadratic factor $(2z^2+z-6)$

$2z^2+z-6=0$

$(2z-3)(z+2)=0$

$z=\frac{3}{2}$ or $z=-2$

$\boxed{z=\frac{3}{2},-2,5\pm i}$

c. sub $a=2, b=1, c=-6$ into (3) and (4):

(3) $c-10b+26a=A$ (4) $-10c+26b=B$

$(-6)-10(1)+26(2)=A$ $-10(-6)+26(1)=B$

$A=36$ $B=86$

$\boxed{A=36,\ B=86}$

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5. The quadratic equation

$$2x^2 - 3x + 5 = 0$$

has roots α and β

Without solving the equation,

(a) write down the value of $(\alpha + \beta)$ and the value of $\alpha\beta$

(1)

(b) determine the value of

(i) $\alpha^2 + \beta^2$

(ii) $\alpha^3 + \beta^3$

(4)

(c) find a quadratic equation which has roots

$$(\alpha^3 - \beta) \text{ and } (\beta^3 - \alpha)$$

giving your answer in the form $px^2 + qx + r = 0$ where p, q and r are integers to be determined.

(5)

a. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

let α, β be roots of quadratic

$x^2 - (\alpha + \beta)x + (\alpha\beta) = 0$

sum of roots : $\alpha + \beta$

product of roots: $\alpha\beta$

$2x^2 - 3x + 5 = 0$ $\div 2$

$x^2 - \frac{3}{2}x + \frac{5}{2} = 0$

$\alpha + \beta = \frac{3}{2}$

$\alpha\beta = \frac{5}{2}$

$\boxed{\alpha + \beta = \frac{3}{2} \quad \alpha\beta = \frac{5}{2}}$

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Question 5 continued

bi. $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= \left(\frac{3}{2}\right)^2 - 2\left(\frac{5}{2}\right)$

$= -\frac{11}{4}$

$\boxed{\alpha^2 + \beta^2 = -\frac{11}{4}}$

ii. $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

$= \left(\frac{3}{2}\right)^3 - 3\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)$

$= -\frac{63}{8}$

$\boxed{\alpha^3 + \beta^3 = -\frac{63}{8}}$

c. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

$x^2 - (\alpha^3 - \beta + \beta^3 - \alpha)x + ((\alpha^3 - \beta) \times (\beta^3 - \alpha)) = 0$

$x^2 - (\alpha^3 + \beta^3 - \alpha - \beta)x + (\alpha^3\beta^3 - \alpha^4 - \beta^4 + \alpha\beta) = 0$

$x^2 - ((\alpha^3 + \beta^3) - (\alpha+\beta))x + ((\alpha\beta)^3 + \alpha\beta - (\alpha^4 + \beta^4)) = 0$

$x^2 - (\alpha^3 + \beta^3 - (\alpha+\beta))x + ((\alpha\beta)^3 + (\alpha\beta) - ((\alpha^2+\beta^2)^2 - 2(\alpha\beta)^2)) = 0$

use ans from part (bii)

use ans from part (bi)

$x^2 - \left(\left(-\frac{63}{8}\right) - \left(\frac{3}{2}\right)\right)x + \left(\left(\frac{5}{2}\right)^3 + \left(\frac{5}{2}\right) - \left(\left(-\frac{11}{4}\right)^2 - 2\left(\frac{5}{2}\right)^2\right)\right) = 0$

$x^2 - \left(-\frac{75}{8}\right)x + \left(\frac{369}{16}\right) = 0$

$x^2 + \frac{75}{8}x + \frac{369}{16} = 0$

$x^2 + \frac{75}{8}x + \frac{369}{16} = 0$ $\times 16$

$16x^2 + 150x + 369 = 0$ ← all integer coefficients

$\boxed{16x^2 + 150x + 369 = 0 \qquad p = 16,\ q = 150,\ r = 369}$

P 7 3 3 3 4 A 0 1 7 3 6

6. The parabola C has equation $y^2 = 36x$

The point $P(9t^2, 18t)$, where $t \neq 0$, lies on C

(a) Use calculus to show that the normal to C at P has equation

$$y + tx = 9t^3 + 18t$$

(4)

(b) Hence find the equations of the two normals to C which pass through the point (54, 0), giving your answers in the form $y = px + q$ where p and q are constants to be determined.

(4)

Given that

- the normals found in part (b) intersect the directrix of C at the points A and B
- the point F is the focus of C

(c) determine the area of triangle AFB

(3)

a. Differentiate hyperbola eq^n w.r.t. x:

$y^2 = 36x$

$2y\left(\frac{dy}{dx}\right) = 36$

$\frac{dy}{dx} = \frac{36}{2y} = \frac{18}{y}$

Sub in point $P(9t^2, 18t)$:

$\left.\frac{dy}{dx}\right|_{(9t^2, 18t)} \quad \frac{18}{(18t)} = \frac{18(1)}{18(t)} = \frac{1}{t}$

$M_{tangent} = \frac{1}{t}$

$M_{normal} = -t$

Set up line eq^n $y - y_1 = m(x - x_1)$ with $m = -t$

$(x_1, y_1) = (9t^2, 18t)$

$y - 18t = -t(x - 9t^2)$

$y - 18t = -tx + 9t^3$

$\boxed{y + tx = 9t^3 + 18t \text{ // shown}}$

Question 6 continued

b. sub in $(54, 0)$ $x = 54, y = 0$ into normal eq^n

$$y + tx = 9t^3 + 18t$$

$$(0) + t(54) = 9t^3 + 18t$$

$$54t = 9t^3 + 18t$$

$$9t^3 - 36t = 0$$

$$9t(t^2 - 4) = 0$$

$$9t(t-2)(t+2) = 0$$

$$t = 0 \quad t = 2 \quad t = -2$$

sub in all t-values into line eq^n:

(1) $t = 0$, $y + (0)x = 9(0)^3 + 18(0)$

$y = 0$

(2) $t = 2$, $y + (2)x = 9(2)^3 + 18(2)$

$y + 2x = 108 \implies y = -2x + 108$

(3) $t = -2$, $y + (-2)x = 9(-2)^3 + 18(-2)$

$y - 2x = -108 \implies y = 2x - 108$

$$\boxed{\begin{aligned} y &= 2x - 108 \\ y &= -2x + 108 \end{aligned}}$$

c. $y^2 = 36x$

$y^2 = 4(9)x \qquad a = 9$

directrix eq^n: $x = -9$

focus: $(9, 0)$

Conics

	Ellipse	Parabola	Hyperbola	Rectangular Hyperbola
Standard Form	$\cancel{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$	$y^2 = 4ax$	$\cancel{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$	$\cancel{xy = c^2}$
Parametric Form	$\cancel{(a\cos\theta, b\sin\theta)}$	$(at^2, 2at)$	$\cancel{(a\sec\theta, b\tan\theta)}$ $\cancel{(\pm a\cosh\theta, b\sinh\theta)}$	$\cancel{\left(ct, \frac{c}{t}\right)}$
Eccentricity	$\cancel{e < 1}$ $\cancel{b^2 = a^2(1 - e^2)}$	$e = 1$	$\cancel{e > 1}$ $\cancel{b^2 = a^2(e^2 - 1)}$	$\cancel{e = \sqrt{2}}$
Foci	$\cancel{(\pm ae, 0)}$	$(a, 0)$	$\cancel{(\pm ae, 0)}$	$\cancel{(\pm\sqrt{2}c, \pm\sqrt{2}c)}$
Directrices	$\cancel{x = \pm\frac{a}{e}}$	$x = -a$	$\cancel{x = \pm\frac{a}{e}}$	$\cancel{x + y = \pm\sqrt{2}c}$
Asymptotes	~~none~~	none	$\cancel{\frac{x}{a} = \pm\frac{y}{b}}$	$\cancel{x = 0, y = 0}$

(1) find points A and B by solving each of the normal line eq^{n}s simultaneously with directrix eq^n.

Question 6 continued

(1) $y = 2x - 108$

(2) $x = -9$

$y = 2(-9) - 108$

$y = -126$

$A(-9, -126)$

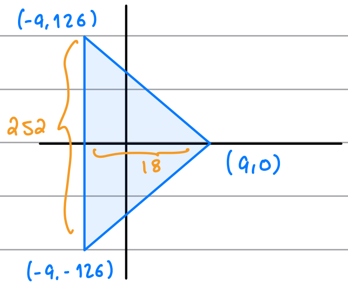
(1) $y = -2x + 108$

(2) $x = -9$

$y = -2(-9) + 108$

$y = 126$

$B(-9, 126)$



$\Delta AFB = \frac{1}{2} \times 18 \times 252$

$= 2268$

$\boxed{\Delta AFB = 2268 \text{ units}^2}$

7.

$$\mathbf{A} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

(a) Determine the matrix $\mathbf{A}^2$

(1)

(b) Describe fully the single geometrical transformation represented by the matrix $\mathbf{A}^2$

(2)

(c) Hence determine the smallest positive integer value of n for which $\mathbf{A}^n = \mathbf{I}$

(1)

The matrix **B** represents a stretch scale factor 4 parallel to the x-axis.

(d) Write down the matrix **B**

(1)

The transformation represented by matrix **A** followed by the transformation represented by matrix **B** is represented by the matrix **C**

(e) Determine the matrix **C**

(2)

The parallelogram P is transformed onto the parallelogram P' by the matrix **C**

(f) Given that the area of parallelogram P' is 20 square units, determine the area of parallelogram P

(2)

a. $A^2 = AA$

$= \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$

$= \begin{bmatrix} (-\frac{\sqrt{3}}{2})(-\frac{\sqrt{3}}{2}) + (-\frac{1}{2})(\frac{1}{2}) & (-\frac{\sqrt{3}}{2})(-\frac{1}{2}) + (-\frac{1}{2})(-\frac{\sqrt{3}}{2}) \\ (\frac{1}{2})(-\frac{\sqrt{3}}{2}) + (-\frac{\sqrt{3}}{2})(\frac{1}{2}) & (\frac{1}{2})(-\frac{1}{2}) + (-\frac{\sqrt{3}}{2})(-\frac{\sqrt{3}}{2}) \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

$\boxed{A^2 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}}$

Question 7 continued

b $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ ← Anticlockwise rotation through θ about O (Given in Formulae booklet)

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

Equate the entries in the matrix: $\cos\theta = \frac{1}{2}$ $\quad \sin\theta = \frac{-\sqrt{3}}{2}$

$$\frac{\sin\theta}{\cos\theta} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

$$\tan\theta = -\sqrt{3}$$

$$\theta = \tan^{-1}(-\sqrt{3})$$

$$\theta = -60°$$

Rotation $-60°$ anticlockwise about the origin

c. A^2 represents a rotation of $-60°$.
A^2 rotated 6 times rotates $-360°$ (I)

$(A^2)^6 =$ rotation $-360° = I$

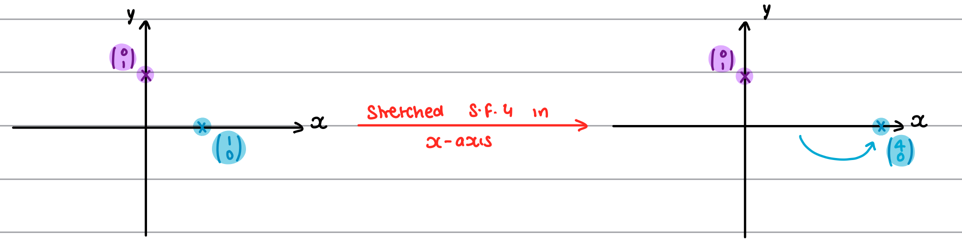
$A^{12} = I$

$\therefore n = 12$

Question 7 continued

d. Start by drawing a coordinate axis and mark coordinates $\binom{1}{0}$ and $\binom{0}{1}$

then transform both of these coordinates by transformation (Stretch S.F. 4 parallel to x-axis)



identity matrix I

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{new matrix represents transformation}} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

rewrite matrix but now replace coordinates with their transformations in this specific order

$$\boxed{B : \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}}$$

e. order: $C = BA$

$$C = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$C = \begin{bmatrix} (4)(-\frac{\sqrt{3}}{2}) + (0)(\frac{1}{2}) & (4)(-\frac{1}{2}) + (0)(-\frac{\sqrt{3}}{2}) \\ (0)(-\frac{\sqrt{3}}{2}) + (1)(\frac{1}{2}) & (0)(-\frac{1}{2}) + (1)(-\frac{\sqrt{3}}{2}) \end{bmatrix}$$

$$\boxed{C = \begin{bmatrix} -2\sqrt{3} & -2 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}}$$

f. $|\det(C)|$ = area scale factor

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\det(X) = (a)(d) - (b)(c)$

$\det(C) = (-2\sqrt{3})(-\frac{\sqrt{3}}{2}) - (-2)(\frac{1}{2})$

$= 4$

$\det(C) = 4$

Question 7 continued

Area S.f. = |det(C)| = |4| = 4

Area of P × 4 = Area of P'

$x \times 4 = 20$

$4x = 20$

$x = 5$

$\boxed{\text{Area of P} = 5 \text{ units}^2}$

(Total for Question 7 is 9 marks)

8. (a) Use the standard results for $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$ to show that for all positive integers n

$$\sum_{r=0}^{n}(r+1)(r+2) = \frac{1}{3}(n+1)(n+2)(n+3)$$

(5)

(b) Hence determine the value of

$$10 \times 11 + 11 \times 12 + 12 \times 13 + \ldots + 100 \times 101$$

(3)

a. $\sum_{r=0}^{n}(r+1)(r+2) = \sum_{r=0}^{n} r^2+3r+2$

in order to use summation results, must start from 1.

∴ find the result of $(r+1)(r+2)$ when $r=0$, then after can use summation results

$\sum_{r=0}^{n} r^2+3r+2 = (r^2+3r+2)\big|_{r=0} + \sum_{r=1}^{n} r^2+3r+2$

$= (0)^2+3(0)+2 + \sum_{r=1}^{n} r^2+3r+2$

$= 2 + \underbrace{\sum_{r=1}^{n} r^2+3r+2}_{\text{solve}}$

$\sum_{r=1}^{n} r^2+3r+2 = \sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + 2\sum_{r=1}^{n} 1$

$= \left[\frac{1}{6}n(n+1)(2n+1)\right] + 3\left[\frac{1}{2}n(n+1)\right] + 2\left[n\right]$

$= \frac{1}{6}n(n+1)(2n+1) + \frac{3}{2}n(n+1) + 2n$

Summations

$\sum_{r=1}^{n} 1 = n$

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ must learn!

$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ in the formulae booklet

substitute back

∴ $\sum_{r=0}^{n} r^2+3r+2 = 2 + \frac{1}{6}n(n+1)(2n+1) + \frac{3}{2}n(n+1) + 2n$

$= \frac{1}{6}n(n+1)(2n+1) + \frac{3}{2}n(n+1) + 2n + 2$

$= \frac{1}{6}n(n+1)(2n+1) + \frac{3}{2}n(n+1) + 2(n+1)$

factor out $\frac{1}{3}(n+1)$

$= \frac{1}{3}(n+1)\left[\frac{1}{2}n(2n+1) + \frac{9}{2}n + 6\right]$

Question 8 continued

$= \frac{1}{3}(n+1)\left[\frac{1}{2}n(2n+1) + \frac{9}{2}n + 6\right]$

$= \frac{1}{3}(n+1)\left[n^2 + \frac{1}{2}n + \frac{9}{2}n + 6\right]$

$= \frac{1}{3}(n+1)\left[n^2 + 5n + 6\right]$

$\boxed{\frac{1}{3}(n+1)(n+2)(n+3) \quad // \text{ shown}}$

b. $\underbrace{10 \times 11}_{\text{LOWER LIMIT}} + 11 \times 12 + 12 \times 13 + \ldots + \underbrace{100 \times 101}_{\text{UPPER LIMIT}}$

$10 \times 11 \equiv (r+1)(r+2) \Rightarrow$ lower limit $= 9$

$100 \times 101 \equiv (r+1)(r+2) \Rightarrow$ upper limit $= 99$

$\sum_{r=9}^{99}(r+1)(r+2) = \sum_{r=0}^{99}(r+1)(r+2) - \sum_{r=0}^{8}(r+1)(r+2)$

use part (a) ans

$= \frac{1}{3}(99+1)(99+2)(99+3) - \frac{1}{3}(8+1)(8+2)(8+3)$

$= 343\,070$

$\boxed{343070}$

P 7 3 3 3 4 A 0 2 9 3 6

9. (i) A sequence of numbers is defined by

$$u_1 = 3$$

$$u_{n+1} = 2u_n - 2^{n+1} \qquad n \geqslant 1$$

Prove by induction that, for $n \in \mathbb{N}$

$$u_n = 5 \times 2^{n-1} - n \times 2^n$$

(5)

(ii) Prove by induction that, for $n \in \mathbb{N}$

$$f(n) = 5^{n+2} - 4n - 9$$

is divisible by 16

(5)

i. (1) Base case: Prove true for $n = 1$

$U_1 = 5 \times 2^{1-1} - 1 \times 2^1$

$U_1 = 3$

$\therefore$ true for $n = 1$

(2) Assume true for $n = k$:

$U_k = 5 \times 2^{k-1} - k \times 2^k$

(3) Prove true for $n = k+1$:

$U_{k+1} = 2U_k - 2^{k+1}$

$= 2\left[5 \times 2^{k-1} - k \times 2^k\right] - 2^{k+1}$ } sub U_k assumed true from step (2)

$= (2)(5)(2^{k-1}) - (2)(k)(2^k) - 2^{k+1}$ } index laws: $a^m \times a^n = a^{m+n}$

$= (5)(2^1)(2^{k-1}) - (k)(2^1)(2^k) - 2^{k+1}$ $\quad \rightarrow 2^1 \times 2^{k-1} = 2^{1+k-1} = 2^k$

$= (5)(2^k) - (k)(2^{k+1}) - 2^{k+1}$ $\quad \rightarrow 2^1 \times 2^k = 2^{1+k}$

$= (5)(2^{k+1-1}) - (k+1)(2^{k+1})$ } rewrite k as $k+1-1$ to match pink and factor out 2^{k+1} for last 2 terms

$= 5 \times 2^{k+1-1} - (k+1) \times 2^{k+1}$

Thinking ahead, sub in $n = k+1$ into U_n – this is what we are trying to prove using (2):

$U_{k+1} = 5 \times 2^{k+1-1} - (k+1) \times 2^{k+1}$

Make a note to the side so we know what we are working towards.

(4) Conclusion

If this result is true for $n = k$, then it is true for $n = k+1$. As the result has been shown to be true for $n = 1$, then the result is true for all n.

Question 9 continued

(1) Base Case: check true for $n=1$:

$f(1) = 5^{1+2} - 4(1) - 9$

$= 112$

$= 7(16)$ ⟵ divisible by 16

$\therefore$ true for $n=1$

(2) Assume true for $n=k$:

$f(k) = 5^{k+2} - 4k - 9$

(3) Inductive step, prove true for $n=k+1$:

$f(k+1) = 5^{k+1+2} - 4(k+1) - 9$

$= 5^{k+3} - 4k - 4 - 9$

$= 5^{k+3} - 4k - 13$

$f(k+1) - f(k) = (5^{k+3} - 4k - 13) - (5^{k+2} - 4k - 9)$

$= (5(5^{k+2}) - 4k - 13) - (5^{k+2} - 4k - 9)$

$= 5(5^{k+2}) - \cancel{4k} - 13 - 5^{k+2} + \cancel{4k} + 9$

$= 4(5^{k+2}) - 4$

$= 4[(5^{k+2}) - 4n - 9] + 16k + 32$

$= 4f(k) + 16k + 32$

$f(k+1) - f(k) = 4f(k) + 16k + 32$

$f(k+1) = 5f(k) + 16(k+2)$ $\therefore$ true for $n=k+1$

(4) Conclusion

If this result is true for $n=k$, then it is true for $n=k+1$. As the result has been shown to be true for $n=2$, then the result is true for all n.